About Knot Theory

The magic of knots

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PRIME Circle Conference, May 13

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Definition of Knots

A *knot* is a loop in space that does not intersect itself anywhere.

- We always work with *projections* of knots, which are curves on a flat surface representing the knots.

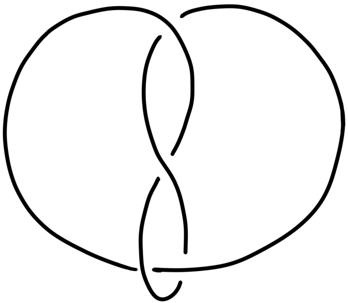

Figure: The figure-eight knot

Definition of Knots: Trivial Knots

Let's give some examples of knots.

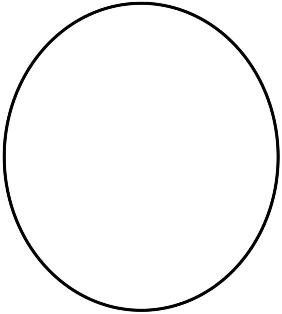

Figure: Unknot

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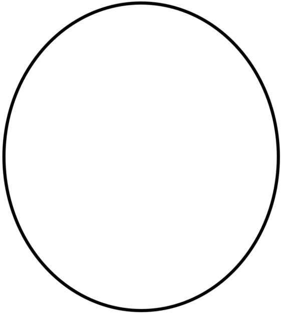


Figure: Unknot

This knot can be also called the *trivial knot*.

Definition of Knots: Nontrival Knots

- The second type of knot are the nontrivial knots. These are the knots which we cannot untie.

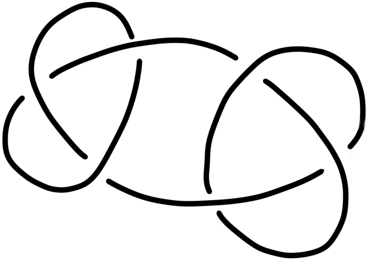


Figure: Double trefoil

Definition of Knots: Nontrival Knots

- The second type of knot are the nontrivial knots. These are the knots which we cannot untie.
- It turns out that any projection of these knots must always have at least three crossings.

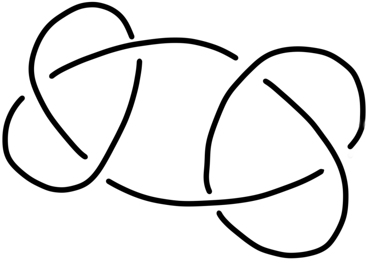
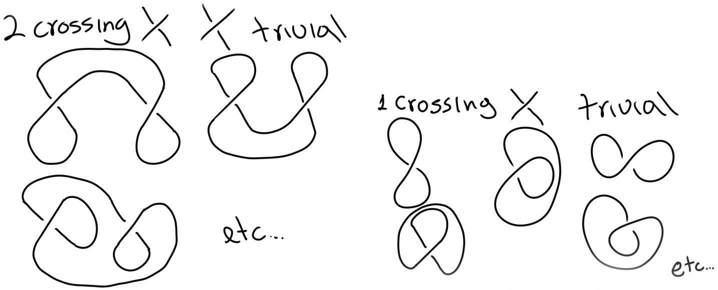


Figure: Double trefoil

We can check all the cases for one crossing and two crossings and they all turn out to be trivial knots.



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Tricolorability

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- If a knot is tricolorable and other knot is not, these knots are different.

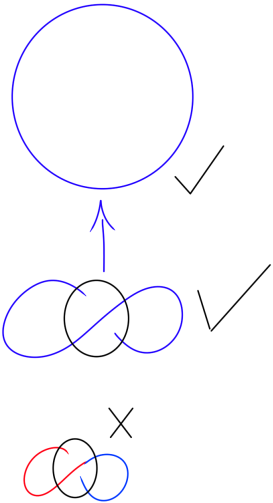
Tricolorability: Rules

The rules of tricolorability are the following:

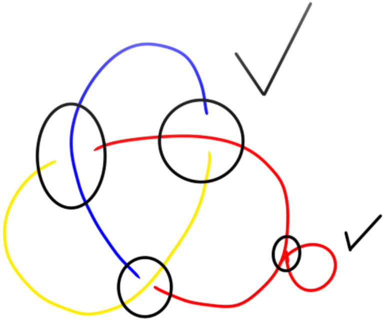
Rules

- We color each arc of the projection of the knot with one of three colors.
- We must use at least two colors.
- Each crossing must have either 1 color or 3 colors, but it cannot have only 2 colors.

Tricolorability: Rules



Tricolorability: Rules



Tricolorability: Examples

Let's make an example about tricolorability.

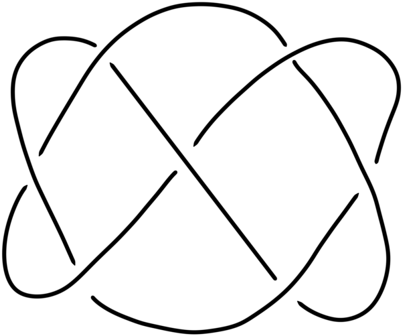


Figure: 7_4 knot

Tricolorability: Examples

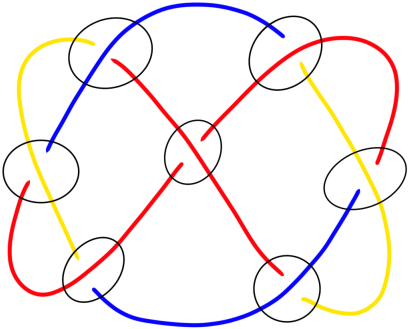


Figure: 7_4 knot is tricolorable

Tricolorability: Conclusion

Conclusion

Thus, we get that the 7_4 knot is tricolorable. This is good. In particular, as 7_4 is tricolorable and we know that the unknot is not tricolorable, this means that these knots are different.

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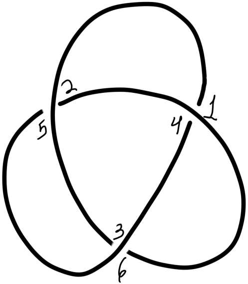
Dowker's Notation

- Dowker's notation is a method we use to encode a knot using numbers.
- We can also use it to produce knots in some situations.
- We use it because it is more common in math to identify things with numbers.

Let's see an example with the trefoil knot.



1	3	5
4	6	2

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- Then, we made a table and put in order all the odd numbers in the top and all the corresponding even number in the bottom.
- We got the following, which we call the *Dowker notation* of this knot:

1	3	5
4	6	2

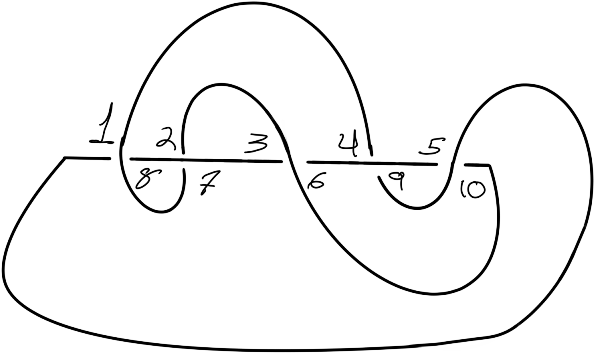
Dowker's Notation: Examples

Now, we will learn how to get a knot from a given Dowker's notation. We will use the information provided by the numbers and with this information we will graph it.

$$\begin{matrix} 1 & 3 & 5 & 7 & 9 \\ 8 & 6 & 10 & 2 & 4 \end{matrix}$$

Dowker's Notation: Examples

1	3	5	7	9
8	6	10	2	4



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Knots and Sticks

What if we tell you that you can make knots with sticks? In the real world, making knots with sticks is a usual way to construct a knot.

- The minimum number of sticks that you need to make a knot is 3 sticks.

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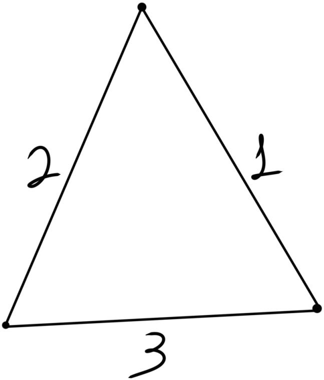

Figure: Unknot with sticks

Knots and Sticks: Formula

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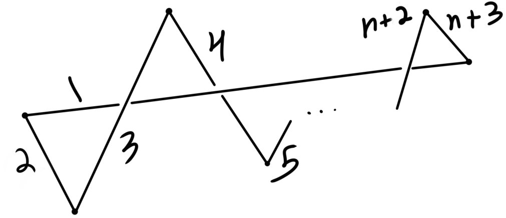
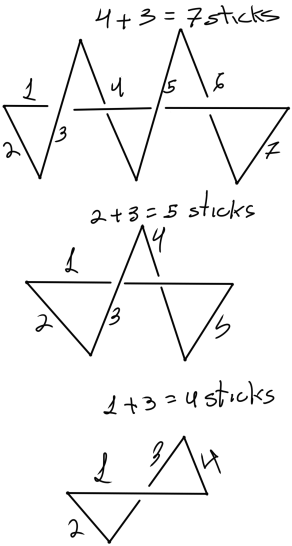


Knots and Sticks: Example

Examples

- If $n = 1$, we can make it with $1 + 3 = 4$ sticks.
- If $n = 2$, we can make it with $2 + 3 = 5$ sticks.
- If $n = 4$, we can make it with $4 + 3 = 7$ sticks.

Knots and Sticks: Examples



Knots and Sticks: Exercise

If you put your hands together, you will be a knot with five sticks. However, can you make a crossing with your arms?

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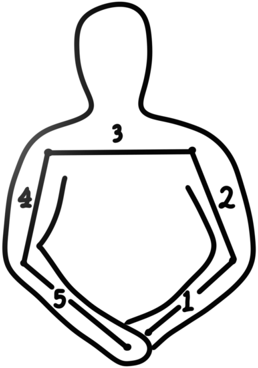

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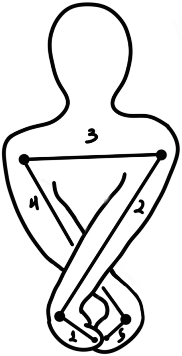

Thank you!
Questions or comments ?